

Imaginaries in bounded pseudo real closed fields

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Abstract

The main result of this paper is that if M is a bounded PRC field then $Th(M)$ eliminates imaginaries in the language of rings expanded by constant symbols.

1 Introduction

A *pseudo algebraically closed field* (PAC field) is a field M such that every absolutely irreducible affine variety defined over M has an M -rational point. The concept of a PAC field was introduced by J. Ax in [1] and has been extensively studied. The above definition of PAC field has an equivalent model-theoretic version: M is existentially closed (in the language of rings) into each regular field extension of M .

A field M is called *bounded* if for any integer n , M has only finitely many extensions of degree n . Hrushovski showed in [8] that if M is a bounded PAC field, and \mathcal{L} is the language of rings expanded by enough constants, then $Th_{\mathcal{L}}(M)$ eliminate imaginaries.

The notion of PAC field has been generalized by Basarab in [2] and then by Prestel in [14] for ordered fields. Prestel calls a field M *pseudo real closed field* (PRC field) if M is existentially closed (in the language of rings) into each regular field extension L to which all orderings of M extend. Remark that if M is a PRC field and has no orderings, then M is a PAC field. PRC fields were extensively studied by L. van den Dries in [15], A. Prestel in [14], M. Jarden in [9], [10], [11], S. Basarab in [4] and [3], and others.

The main result in this paper is a generalization to PRC bounded fields of elimination of imaginaries for PAC fields.

As corollary of the elimination of imaginaries and the fact that the algebraic closure in the sense of model theory defines a pregeometry we obtain (Theorem 4.10) that the complete theory of a bounded PRC field is superrosy of U^b -rank 1.

The organization of the paper is as follows: In section 2 we give the required preliminaries on pseudo real closed fields and we fix a complete theory T of a bounded PRC field, where we enrich the language adding constants for an elementary submodel. In section 3 for $n \geq 1$, we define the theory VO_n in a multi-sorted language \mathcal{L}_n^* . To each model of the theory T we associate a model of VO_n (Remark 3.2). This result is an important tool in the proof of elimination of imaginaries for bounded PRC fields. We show quantifier elimination and elimination of imaginaries for the theory VO_n . Finally in section 4 we prove the elimination of imaginaries for bounded PRC fields (Theorem 4.8).

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2 Pseudo real closed fields

In this section we give the required preliminaries in pseudo real closed fields.

Notations and Conventions 2.1. If M is a model of an \mathcal{L} -theory T and $A \subseteq M$, then $\mathcal{L}(A)$ denotes the set of \mathcal{L} -formulas with parameters in A . If \bar{a} is a tuple of M , we denote by $\text{tp}_{\mathcal{L}}^M(a/A)$ ($\text{qftp}_{\mathcal{L}}^M(a/A)$) the set of $\mathcal{L}(A)$ -formulas (quantifier-free $\mathcal{L}(A)$ -formulas) φ , such that $M \models \varphi(a)$. Denote by $\text{acl}_{\mathcal{L}}^M$ and $\text{dcl}_{\mathcal{L}}^M$ the model theoretic algebraic and definable closure in M . We omit M or \mathcal{L} when the structure or the language is clear. We denote by $\mathcal{L}_{\mathcal{R}}$ the language of rings. **All fields considered in this paper will have characteristic zero.**

Definition 2.2. Let M, N be fields such that $M \subseteq N$.

- (1) The extension N/M is called *totally real* if each order on M extends to some order on N .
- (2) We say that N is a *regular extension* of M if $N \cap M^{\text{alg}} = M$.

Fact 2.3. [14, Theorem 1.2] *For a field M the following are equivalent:*

- (1) M is existentially closed (relative to $\mathcal{L}_{\mathcal{R}}$) in every totally real regular extension N of M .
- (2) For every absolutely irreducible variety V defined over M , if V has a simple \overline{M}^r -rational point for every real closure \overline{M}^r of M , then V has an M -rational point.

Definition 2.4. A field M that satisfies the conditions of Fact 2.3 is *pseudo real closed (PRC)*. By [14, Theorem 4.1] we can axiomatize the class of *PRC* fields in $\mathcal{L}_{\mathcal{R}}$. Remark that the class of *PRC* fields contains the class of *PAC* fields and the class of real closed fields (*RCF* fields).

In the case when M admits only a finite number of orderings this already implies that M is existentially closed in N even in the language augmented by predicates for each order $<$ of M [14, Theorem 1.7].

Fact 2.5. [10, Proposition 1.2] *Let M be a PRC field. Then:*

- (1) If $<$ is an order on M , then M is $<$ -dense in \overline{M}^r , where \overline{M}^r is the real closure of M with respect to the order $<$.
- (2) If $<_i$ and $<_j$ are different orders on M , then $<_i$ and $<_j$ induce different topologies.

2.6. Approximation Theorem [15, 1.7] Let M be a field and τ_1, \dots, τ_n different topologies on M induced by orders. For each $i \in \{1, \dots, n\}$, let U_i be a non-empty τ_i -open subset of M .

Then $\bigcap_{i=1}^n U_i \neq \emptyset$.

2.1 Bounded pseudo real closed fields

2.7. Bounded fields: If M is a bounded field and M^* an elementary extension of M (in a language containing the language of rings) then the restriction map: $G(M^*) \rightarrow G(M)$ is an isomorphism [5, Lemma 1.22].

2.8. Notation: We fix a bounded *PRC* field K , which is not real closed and a countable elementary substructure K_0 of K . Then $G(K_0) \cong G(K)$ and $K_0^{alg}K = K^{alg}$. Since K is bounded there exists $n \in \mathbb{N}$ such that K has exactly n distinct orders (see Remark 3.2 of [12]). Then we can view K as a structure of the form $(K, <_1, \dots, <_n)$, where $\{<_1, \dots, <_n\}$ are all the different orders on the field K .

In this paper we will work over K_0 , thus we denote by \mathcal{L} the language of rings with constant symbols for the elements of K_0 , $\mathcal{L}^{(i)} := \mathcal{L} \cup \{<_i\}$ and $\mathcal{L}_n := \mathcal{L} \cup \{<_1, \dots, <_n\}$. We let $T := Th_{\mathcal{L}_n}(K)$.

If $n = 0$, then K is a *PAC* field and by Corollary 3.1 of [8] $Th_{\mathcal{L}}(M)$ has elimination of imaginaries. Thus we will suppose that $n \geq 1$.

Fact 2.9. [12] *Let $(M, <_1, \dots, <_n)$ be a model of T .*

- (1) *For all $i \in \{1, \dots, n\}$ the order $<_i$ is \exists -definable in the language \mathcal{L} , and $Th_{\mathcal{L}}(M)$ is model complete.*
- (2) *If A is a subfield of M containing K_0 , then $A^{alg} \cap M = acl_{\mathcal{L}_n}^M(A) = dcl_{\mathcal{L}_n}^M(A) = acl_{\mathcal{L}}^M(A) = dcl_{\mathcal{L}}^M(A)$.*

3 The theory VO_n

Definition 3.1. Fix $n \in \mathbb{N}$, $n \geq 1$. Let \mathcal{L}_n^* be the $(n+1)$ -sorted language consisting of $n+1$ sorts $\{R_0, \dots, R_n\}$, n binary relation symbols $\{<_1, \dots, <_n\}$, with $<_i$ on the sort R_i , a constant symbol 0 of sort R_0 , and $2n$ function symbols $\{f_1, \dots, f_n, g_1, \dots, g_n\}$, where $f_i : R_0 \rightarrow R_i$ and $g_i : R_i \rightarrow R_0$. Let VO_n be the \mathcal{L}_n^* -theory axiomatized by:

- (1) $<_i$ defines a dense linear order without endpoints on R_i , for $i \in \{1, \dots, n\}$,
- (2) $f_i : R_0 \rightarrow R_i$ is an injective function, for $i \in \{1, \dots, n\}$,
- (3) $g_i(x) := \begin{cases} f_i^{-1}(x) & \text{if } x \in f_i(R_0); \\ 0 & \text{otherwise.} \end{cases}$
- (4) $f_i(R_0)$ is $<_i$ -dense and $<_i$ -co-dense in R_i , for $i \in \{1, \dots, n\}$,
- (5) If $a_i, b_i \in R_i$ are such that $a_i <_i b_i$, then there exists $x_0 \in R_0$ such that $a_i <_i f_i(x_0) <_i b_i$, for all $i \in \{1, \dots, n\}$.

Notation:

(1) If $a_i, b_i \in R_i \cup \{\pm\infty\}$ and $a_i <_i b_i$, denote by $(a_i, b_i)_i := \{x \in R_i : a_i <_i x <_i b_i\}$ and by $f_i^{-1}(a_i, b_i)_i := \{x \in R_0 : f_i(x) \in (a_i, b_i)_i\}$.

(2) If M is a model of VO_n and $A \subseteq M$, denote by $\langle A \rangle$ the \mathcal{L}_n^* -substructure generated by A .

Remark 3.2. If M is a model of T (see 2.8), we associate to M a model of VO_n as follows: For each $i \in \{1, \dots, n\}$ we let $M^{(i)}$ be a real closure of M for the order $<_i$. We define $R_0 = M$, $R_i = M^{(i)}$ with its natural order $<_i$, $f_i : M \rightarrow M^{(i)}$ the natural inclusion and g_i its “inverse”. Axioms 1, 2 and 3 are clearly true. Axiom 5 is true by the Approximation Theorem (2.6) and Fact 2.5. For axiom 4, by Fact 2.5(1), M is $<_i$ -dense in $M^{(i)}$. To see that M is co-dense in $M^{(i)}$, suppose by contradiction that there exists an $<_i$ -interval I such that $I \subseteq M$. Let $\alpha \in M^{(i)} \setminus M$ and define the $<_i$ -interval $J := \{x + \alpha : x \in I\}$. Then $J \subset (M^{(i)} \setminus M)$, which contradicts M being dense in $M^{(i)}$.

Observe that if $a_i, b_i \in M^{(i)} \cup \{\pm\infty\}$ and $I = \bigcap_{i=1}^n ((a_i, b_i)_i \cap M)$, then in the corresponding model of VO_n , the set I is \mathcal{L}_n^* -definable by the formula $\bigcap_{i=1}^n f_i^{-1}(a_i, b_i)_i$.

Theorem 3.3. *The theory VO_n is \aleph_0 -categorical, has quantifier elimination in \mathcal{L}_n^* and if M is a model of VO_n and $A \subseteq M$, then $\text{acl}_{\mathcal{L}_n^*}(A) = \text{dcl}_{\mathcal{L}_n^*}(A) = \langle A \rangle = \bigcup_{a \in A} \langle a \rangle$.*

Proof. Let $M = (M_0, \dots, M_n)$ and $N = (N_0, \dots, N_n)$ be countable models of VO_n . Let $\mathcal{I}(M, N)$ be the family of partial isomorphisms with finite domain of M to N . Let $g \in \mathcal{I}(M, N)$, $A = \text{dom}(g)$ and $B = g(A)$. Let $b \in M \setminus A$; we need to find $c \in M$ such that if $\tilde{g} : A \cup \{b\} \rightarrow B \cup \{c\}$ is defined by $\tilde{g}|_A = g$ and $\tilde{g}(b) = c$, then $\tilde{g} \in \mathcal{I}(M, N)$.

As g extends uniquely to an isomorphism from $\langle A \rangle$ to $\langle B \rangle$, we can suppose that $A = \langle A \rangle$ and $B = \langle B \rangle$.

Case 1: $b \in M_0$.

Observe that $\langle A, b \rangle = A \cup \{b, f_i(b) : 1 \leq i \leq n\}$. For each $i \in \{1, \dots, n\}$, let I^i be the smallest open $<_i$ -interval containing $f_i(b)$ with extremities in $(A \cap M_i) \cup \{\pm\infty\}$. Let $J^i \subseteq N_i$ be the interval obtained by applying g to the extremities of I^i . By axiom 5 there exists $c \in N_0$ such that $f_i(c) \in J^i$ for all $1 \leq i \leq n$; define $g(b) = c$.

Case 2: There exists $i > 0$ such that $b \in M_i$.

There exists a smallest open $<_i$ -interval I^i with extremities in $(A \cap M_i) \cup \{\pm\infty\}$, such that $b \in I^i$. Let $J^i \subseteq N_i$ be the interval obtained by applying g to the extremities of I^i . We have two possibilities:

- (1) $b \in f_i(M_0)$: apply Case 1 to $g_i^{-1}(b) = b'$, observe that $\langle A, b \rangle = A \cup \{b', f_j(b') : 1 \leq j \leq n\}$.
- (2) $b \notin f_i(M_0)$: as $f_i(N_0)$ is $<_i$ -co-dense in N_i , there is $c \in J^i \setminus f_i(N_0)$. Define $g(b) = c$. Then $\langle A, b \rangle = A \cup \{b\}$.

This shows quantifier elimination and \aleph_0 -categoricity. The assertion about the algebraic and definable closures is clear. \square

Definition 3.4. Let (M_0, \dots, M_n) be a model of VO_n .

- (1) A set of the form $I = \bigcap_{i=1}^n f_i^{-1}(I^i)$, with I^i a non-empty $<_i$ -open interval in M_i is called a

multi-interval. Observe that if a multi-interval $I = \bigcap_{i=1}^n f_i^{-1}(I^i)$ is definable over A , then by quantifier elimination (Theorem 3.3) each I^i has its extremities in $(\langle A \rangle \cap M_i) \cup \{\pm\infty\}$. We call the *set of extremities of the multi-interval I* the set of extremities of I^i , for all $i \in \{1, \dots, n\}$.

- (2) Let $E \subseteq M_0$. We say that E is *multi-open* if for each $e \in E$, there exists a multi-interval I such that $e \in I$ and $I \subseteq E$.

- (3) Let $E \subseteq M_0$ and $e \in E$. Let $I = \bigcap_{i=1}^n f_i^{-1}(I^i)$ be a multi-interval such that $e \in I$ and $I \subseteq E$.

We say that I is the *maximal multi-interval in E containing e* if for all $m \in \{1, \dots, n\}$, I^m is the maximal $<_m$ -interval with the property that:

- (a) $f_m(e) \in I^m$,
- (b) There are J^{m+1}, \dots, J^n intervals in M_{m+1}, \dots, M_n respectively, such that:
 - i. $f_j(e) \in J^j$, for all $m+1 \leq j \leq n$,
 - ii. $\bigcap_{l=1}^m f_l^{-1}(I^l) \cap \bigcap_{j=m+1}^n f_j^{-1}(J^j) \subseteq E$.

Remark 3.5. Observe that if $X(e) = \bigcap_{i=1}^n f_i^{-1}(X^i(e))$ and $Y(e) = \bigcap_{i=1}^n f_i^{-1}(Y^i(e))$ are maximal multi-intervals in E containing e , then for all $i \in \{1, \dots, n\}$, $X^i(e) = Y^i(e)$: It is clear using maximality that $X^1(e) = Y^1(e)$; by induction suppose that $X^1(e) = Y^1(e), \dots, X^m(e) = Y^m(e)$, using maximality again we obtain that $X^{m+1}(e) = Y^{m+1}(e)$.

Lemma 3.6. Let $M = (M_0, \dots, M_n)$ be a model of VO_n and $A \subseteq M$ finite. Let $E \subseteq M_0$ be multi-open $\mathcal{L}_n^*(A)$ -definable, and let $e \in E$. Then there exists a maximal multi-interval $X(e)$ in E containing e and its extremities are in $\langle A \rangle \cup \{\pm\infty\}$.

Proof. Let $Y^1(e)$ be the set of open $<_1$ -intervals J^1 satisfying:

- (1) $f_1(e) \in J^1$.
- (2) There exist J^2, \dots, J^n open intervals in M_2, \dots, M_n respectively such that:
 - (a) $f_j(e) \in J^j$, for all $2 \leq j \leq n$,
 - (b) $\bigcap_{j=1}^n f_j^{-1}(J^j) \subseteq E$.

Observe that if $J^1, L^1 \in Y^1(e)$ then $J^1 \cup L^1 \in Y^1(e)$.

Claim. $Y^1(e)$ has a maximal element.

Proof. Since $e \in E$ and E is multi-open, using axiom 5 we can find $e_1, e_2 \in M_0$, such that for all $i \leq n$, $f_i(e_1) <_i f_i(e_2)$, and $e \in \bigcap_{i=1}^n f_i^{-1}(f_i(e_1), f_i(e_2))_i \subseteq E$. Then $(f_1(e_1), f_1(e_2))_1 \in Y^1(e)$.

Define $X^1(e, e_1, e_2) := \{x \in M_1 : \exists a, b \in M_1 (x \in (a, b)_1 \wedge f_1(e) \in (a, b)_1 \wedge f_1^{-1}(a, b)_1 \cap \bigcap_{i=2}^n f_i^{-1}(f_i(e_1), f_i(e_2))_i \subseteq E)\}$.

Observe that $X^1(e, e_1, e_2)$ is definable, $(f_1(e_1), f_1(e_2))_1 \subseteq X^1(e, e_1, e_2)$ and that it is connected for the $<_1$ -topology: if $x, y \in X^1(e, e_1, e_2)$ and $x <_1 y$ then $(x, y)_1 \subseteq X^1(e, e_1, e_2)$. This implies by quantifier elimination (Theorem 3.3) that $X^1(e, e_1, e_2)$ is an $<_1$ -interval, so that $X^1(e, e_1, e_2) \in Y^1(e)$.

Since $X^1(e, e_1, e_2)$ is definable with parameters in $A \cup \{e, e_1, e_2\}$, its extremities are in $dcl_{\mathcal{L}_n^*}(A \cup \{e, e_1, e_2\}) \cup \{\pm\infty\}$. So by Theorem 3.3, its extremities are in $(\langle A \rangle \cap M_1) \cup \{f_1(e), f_1(e_1), f_1(e_2)\} \cup \{\pm\infty\}$. As $f_1(e) \in (f_1(e_1), f_1(e_2))_1 \subseteq X^1(e, e_1, e_2)$, it cannot be one of the extremities. Thus $X^1(e, e_1, e_2)$ has its extremities in $(\langle A \rangle \cap M_1) \cup \{f_1(e_1), f_1(e_2)\} \cup \{\pm\infty\}$.

Let $e'_1, e'_2 \in M_0$ be such that $(f_i(e'_1), f_i(e'_2))_i \subsetneq (f_i(e_1), f_i(e_2))_i$ for all $i \leq n$, and $e \in \bigcap_{i=1}^n f_i^{-1}(f_i(e'_1), f_i(e'_2))_i \subseteq E$. Then $X^1(e, e_1, e_2) \subseteq X^1(e, e'_1, e'_2)$.

As before we obtain that $X^1(e, e'_1, e'_2)$ is an $<_1$ -open interval with extremities in $(\langle A \rangle \cap M_1) \cup \{f_1(e'_1), f_1(e'_2)\} \cup \{\pm\infty\}$ and that $X^1(e, e'_1, e'_2) \in Y^1(e)$. But as $f_1(e'_1), f_1(e'_2) \in (f_1(e_1), f_1(e_2))_1 \subseteq X^1(e, e_1, e_2) \subseteq X^1(e, e'_1, e'_2)$, then $f_1(e'_1), f_1(e'_2)$ cannot be the extremities of $X^1(e, e'_1, e'_2)$. This implies that $X^1(e, e'_1, e'_2)$ has its extremities in $(\langle A \rangle \cap M_1) \cup \pm\{\infty\}$.

We have shown that if $e'_1, e'_2 \in M_0$ are such that $(f_i(e'_1), f_i(e'_2))_i \subsetneq (f_i(e_1), f_i(e_2))_i$, then $X(e, e'_1, e'_2)$ is $\mathcal{L}_n^*(A)$ -definable and that as the multi-intervals $(e'_1, e'_2)_i$ decrease, the sets $X^1(e, e_1, e_2)$ increase. By \aleph_0 -categoricity and the fact that $\langle A \rangle$ is finite, the sets $X^1(e, e'_1, e'_2)$ stabilize for some value of e'_1, e'_2 and so $Y^1(e)$ has a maximal element $X^1(e)$, which has its extremities in $\langle A \rangle \cup \{\pm\infty\}$. □

By induction suppose that we have already defined $Y^1(e), \dots, Y^{l-1}(e)$, for $l \leq n$, with $X^1(e), \dots, X^{l-1}(e)$ the maximal element of $Y^1(e), \dots, Y^{l-1}(e)$ respectively.

Let $Y^l(e)$ be the set of $<_l$ -intervals J^l satisfying:

- (1) $f_l(e) \in J^l$,
- (2) There exist J^{l+1}, \dots, J^n intervals in M_{l+1}, \dots, M_n respectively such that:
 - (a) $f_j(e) \in J^j$, for all $l+1 \leq j \leq n$,
 - (b) $\bigcap_{j=1}^{l-1} f_j^{-1}(X^j(e)) \cap \bigcap_{j=l}^n f_j^{-1}(J^j) \subseteq E$.

Reasoning exactly as for $l = 1$, and defining

$$X^l(e, e_1, e_2) := \{x \in M_l : \exists a, b \in M_l (x \in (a, b)_l \wedge f_l(e) \in (a, b)_l \wedge \bigcap_{i=1}^{l-1} X^i(e) \cap (a, b)_l \cap \bigcap_{i=l+1}^n f_i^{-1}(f_i(e_1), f_i(e_2))_i \subseteq E\},$$

we find that $Y^l(e)$ has a maximal element $X^l(e)$, which is $\mathcal{L}_n^*(A)$ -definable.

Let $X(e) := \bigcap_{i=1}^n f_i^{-1}(X^i(e))$. Then $X(e)$ is the maximal multi-interval in E containing e , and it is $\mathcal{L}_n^*(A)$ -definable. \square

3.7. Canonical decomposition:

Let $M = (M_0, \dots, M_n)$ be a model of VO_n and $E \subseteq M_0$. Let $B \subseteq \mathbb{N}$ and $E_0 \subseteq E$ be finite sets and for all $j \in B$, let I_j be a multi-interval. We say that $\bigcup_{j \in B} I_j \cup E_0$ is a *canonical decomposition* of E if:

- (1) $E = \bigcup_{j \in B} I_j \cup E_0$,
- (2) for all $e \in E_0$, there is no multi-interval I containing e such that $I \subseteq E$,
- (3) for all $e \in E \setminus E_0$, there exists $j \in B$ such that I_j is the maximal multi-interval in E containing e ,
- (4) for all $j \in B$ there exists $e \in E \setminus E_0$ such that I_j is the maximal multi-interval in E containing e .

Theorem 3.8. *Let $M = (M_0, \dots, M_n)$ be a model of VO_n and $A \subseteq M$ finite. Let $E \subseteq M_0$ be $\mathcal{L}_n^*(A)$ -definable. Then there exists a unique canonical decomposition of E and its extremities are in $\langle A \rangle \cup \{\pm\infty\}$.*

Proof. Define $\tilde{E} := \{x \in E : \text{there exists a multi-interval } I \text{ such that } x \in I \text{ and } I \subseteq E\}$ and $E_0 := E \setminus \tilde{E}$. Observe that \tilde{E} and E_0 are $\mathcal{L}_n^*(A)$ -definable, that \tilde{E} is multi-open and that $E = \tilde{E} \cup E_0$. Using quantifier elimination (Theorem 3.3) and the fact that the negation of an atomic formula is a disjunction of atomic formulas we obtain that E_0 is a finite set, defined by disjunctions of equalities.

As \tilde{E} is multi-open, by Lemma 3.6 for each $e \in \tilde{E}$ there exists $X(e) = \bigcap_{i=1}^n f_i^{-1}(X^i(e))$, the maximal multi-interval in \tilde{E} containing e , and its extremities are in $\langle A \rangle \cup \{\pm\infty\}$.

Since $\langle A \rangle$ is finite then $\{X(e) : e \in \tilde{E}\}$ is finite. Let $B \subset \tilde{E}$ be finite such that $\{X(e) : e \in \tilde{E}\} = \{X(e) : e \in B\}$. Then $\tilde{E} = \bigcup_{e \in B} X(e)$.

Therefore $\bigcup_{e \in B} X(e) \cup E_0$ is a canonical decomposition of E and its extremities are in $\langle A \rangle \cup \{\pm\infty\}$. The uniqueness is clear by Remark 3.5. \square

Remark 3.9. The uniqueness in Theorem 3.8 implies that if $E \subseteq M_0$ is definable with parameters in $\langle A \rangle$ and also with parameters in $\langle B \rangle$, then the canonical decomposition of E is definable with parameters in $\langle A \rangle \cap \langle B \rangle$.

Theorem 3.10. VO_n has elimination of imaginaries in the language \mathcal{L}_n^* .

Proof. Let $M = (M_0, \dots, M_n)$ be a model of VO_n .

Claim. VO_n has unary elimination of imaginaries:

Proof. Let $A \subseteq M$ be finite and let E be an $\mathcal{L}_n^*(A)$ -definable set. We have two cases:

Case 1: $E \subseteq M_0$.

By Theorem 3.8 there exists a canonical decomposition of E . Let \bar{c} be the set of finite points and extremities of the multi-intervals in the canonical decomposition of E . By Remark 3.9, \bar{c} is the code of the set E .

Case 2: $E \subseteq M_i$ for some $i > 0$.

Define $E_1 := E \cap f_i(M_0)$, $E_2 := E \cap (M_i \setminus f_i(M_0))$. Observe that E_1 and E_2 are $\mathcal{L}_n^*(A)$ -definable and that E is the disjoint union of E_1 and E_2 . Since $g_i(E_1) \subseteq M_0$, by Case 1 $g_i(E_1)$ is coded by some tuple $c_1 \in \langle A \rangle$.

By quantifier elimination (Theorem 3.3), E_2 is defined by a boolean combination of formulas of the form: $a <_i x$, $a = x$, $b <_j f_j g_i(x)$, $b = f_j g_i(x)$ with $a \in M_i \cap \langle A \rangle$, $b \in M_j \cap \langle A \rangle$. Since $E_2 \subseteq M_i \setminus f_i(M_0)$, then $f_j(g_i(x)) = f_j(0)$ for all $x \in E_2$. This implies that E_2 is defined by a formula $\psi(x) \wedge g_i(x) = 0$, where $\psi(x)$ is a finite union of points and disjoint intervals with extremities in $(\langle A \rangle \cap M_i) \cup \{\pm\infty\}$. Then the set c_2 of finite points and extremities of these intervals is the code of the set E_2 , and $c := (c_1, c_2)$ is the code of E . \square

By Remark 3.2.2 of [7] it is enough to show that every definable unary function (with parameters) is encoded in M . Let $i, j \in \{0, \dots, n\}$ and $h : M_i \rightarrow M_j$ be an $\mathcal{L}_n^*(A)$ -definable function with $A \subseteq M$ finite.

Let $B := \{x : h(x) \in \langle x \rangle\}$. Observe that $h(M_i \setminus B)$ is finite: Let $x \in M_i \setminus B$; since h is $\mathcal{L}_n^*(A)$ -definable, then $h(x) \in dcl(Ax) = \langle A \rangle \cup \langle x \rangle$. Since $x \notin B$ then $h(x) \notin \langle x \rangle$, and $h(x) \in \langle A \rangle \cap M_j$. As $\langle A \rangle$ is finite, $h(M_i \setminus B) \subseteq \langle A \rangle$ is finite.

Let $m \in \mathbb{N}$ and $\{a_1, \dots, a_m\} \in \langle A \rangle \cap M_j$ be such that $h(M_i \setminus B) = \{a_1, \dots, a_m\}$. Let $X_l := \{x \in M_i : h(x) = a_l\}$; by unary elimination the set X_l is coded in M by a tuple $\ulcorner X_l \urcorner$.

The function $h|_B$ is also coded in M : Let $X_0 := \{x \in M_i : h(x) \in \langle 0 \rangle\} \subseteq B$; by unary elimination the set X_0 is coded in M by a tuple $\ulcorner X_0 \urcorner$. If $D := B \setminus X_0$, then

$$h|_D = \begin{cases} id & \text{if } i = j = 0, \\ f_j g_i & \text{if } i, j > 0, \\ f_j & \text{if } i = 0, j > 0, \\ g_j & \text{if } i > 0, j = 0. \end{cases}$$

Let $d := (\ulcorner X_0 \urcorner, \ulcorner X_1 \urcorner, \dots, \ulcorner X_m \urcorner, a_1, \dots, a_m) \subseteq M$. Then d is the code in M of h . \square

4 Elimination of Imaginaries in bounded PRC fields

Definition 4.1. Let $(M, <_1, \dots, <_n)$ be a model of T (see 2.8). Denote by $M^{(i)}$ a fixed real closure of M with respect to $<_i$.

- (1) A subset of M of the form $I = \bigcap_{i=1}^n (I^i \cap M)$ with I^i a non-empty $<_i$ -open interval in $M^{(i)}$ is called a *multi-interval*. Observe that by 2.6 (Approximation Theorem) and Fact 2.5 (1) every multi-interval is non empty.
- (2) A definable subset S of M is called *multi-open* if for each $x \in S$, there exist a multi-interval I such that $x \in I$ and $I \subseteq S$.
- (3) A definable subset S of a multi-interval $I = \bigcap_{i=1}^n (I^i \cap M)$ is called *multi-dense* in I if for any multi-interval $J \subseteq I$, $J \cap S \neq \emptyset$. Note that multi-density implies $<_i$ -density in I^i , for all $i \in \{1, \dots, n\}$.

Remark 4.2. Let $(M, <_1, \dots, <_n)$ be a model of T . Let $i \in \{1, \dots, n\}$ and $a \in M^{(i)} \setminus M$ such that $a \in \text{acl}^{M^{(i)}}(c)$, with c a tuple in M . Then $A = \{x \in M : x <_i a\}$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}(c)$ -formula.

Proof. By quantifier elimination of the theory of real closed fields (RCF) and the fact that $\text{acl}^{M^{(i)}} = \text{dcl}^{M^{(i)}}$, we can find a quantifier-free $\mathcal{L}^{(i)}$ -formula $\phi(x, c)$, such that $M^{(i)} \models \forall x (x <_i a \leftrightarrow \phi(x, c))$. Then $x \in A$ if and only if $M \models \phi(x, c)$. \square

Fact 4.3. [12, Theorem 3.13] Let $(M, <_1, \dots, <_n)$ be a model of T and $A \subseteq M$. Let $S \subseteq M$ be an $\mathcal{L}_n(A)$ -definable set. Then there are a finite set $S_0 \subseteq S$, $m \in \mathbb{N}$ and I_1, \dots, I_m , with $I_j = \bigcap_{i=1}^n (I_j^i \cap M)$ a multi-interval for all $j \in \{1, \dots, m\}$ such that:

- (1) $S \subseteq \bigcup_{j=1}^m I_j \cup S_0$,
- (2) $\{x \in I_j : x \in S\}$ is multi-dense in I_j for all $1 \leq j \leq m$,
- (3) $I_j^i \subseteq M^{(i)}$ has its extremities in $\text{dcl}_{\mathcal{L}^{(i)}}^{M^{(i)}}(A) \cup \{\pm\infty\}$ for all $1 \leq j \leq m$ and $1 \leq i \leq n$,
- (4) the set $I_j^i \cap M$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}(A)$ -formula, for all $1 \leq j \leq m$ and $1 \leq i \leq n$.

Proposition 4.4. Let $(M, <_1, \dots, <_n)$ be a model of T and $A, B \subseteq M$. Let $S \subseteq M$, \mathcal{L}_n -definable with parameters in A and also in B . Then there are a finite set $S_0 \subseteq S$, $m \in \mathbb{N}$ and I_1, \dots, I_m , with $I_j := \bigcap_{i=1}^n (I_j^i \cap M)$ a multi-interval such that:

- (1) $S \subseteq \bigcup_{j=1}^m I_j \cup S_0$,
- (2) $\{x \in I_j : x \in S\}$ is multi-dense in I_j , for all $1 \leq j \leq m$,
- (3) the set $I_j^i \cap M$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}$ -formula with parameters in $\text{acl}(A) \cap \text{acl}(B)$.

Proof. Define $\tilde{S} := \{x \in M : \text{there exists a multi-interval } I \text{ such that } x \in I \text{ and } I \cap S \text{ is multi-dense in } I\}$. Since S is $\mathcal{L}_n(A)$ -definable and also $\mathcal{L}_n(B)$ -definable, then \tilde{S} is $\mathcal{L}_n(A)$ -definable and also $\mathcal{L}_n(B)$ -definable.

By Fact 4.3 there exists a finite set $S_0 \subseteq S$ such that $S \subseteq \tilde{S} \cup S_0$. As \tilde{S} is multi-open and $\mathcal{L}_n(A)$ -definable, using Fact 4.3 there exists $r_1 \in \mathbb{N}$ and multi-intervals $I_j = \bigcap_{i=1}^n (I_j^i \cap M)$, for all

$1 \leq j \leq r_1$ such that: $\tilde{S} = \bigcup_{j=1}^{r_1} I_j$, and for all $1 \leq i \leq n$, $1 \leq j \leq r_1$, I_j^i has its extremities in $dcl_{\mathcal{L}^{(i)}}^{M^{(i)}}(A) \cup \{\pm\infty\}$.

Similarly, as \tilde{S} is also $\mathcal{L}_n(B)$ -definable, there exists $r_2 \in \mathbb{N}$ and multi-intervals $J_j = \bigcap_{i=1}^n (J_j^i \cap M)$ for all $1 \leq j \leq r_2$ such that: $\tilde{S} = \bigcup_{j=1}^{r_2} J_j$, and for all $1 \leq i \leq n$, $1 \leq j \leq r_2$, J_j^i has its extremities in $dcl_{\mathcal{L}^{(i)}}^{M^{(i)}}(B) \cup \{\pm\infty\}$.

Let \tilde{A} be the set of extremities of I_j^i , for $1 \leq i \leq n$ and $1 \leq j \leq r_1$, and let \tilde{B} be the set of extremities of J_j^i , for $1 \leq i \leq n$ and $1 \leq j \leq r_2$.

Consider the \mathcal{L}_n^* -structure $\tilde{M} = (M, M^{(1)}, \dots, M^{(n)})$ associated to M (see Remark 3.2). Observe that \tilde{S} is $\mathcal{L}_n^*(\tilde{A})$ -definable and also $\mathcal{L}_n^*(\tilde{B})$ -definable in \tilde{M} . Then by Theorem 3.8 and Remark 3.9 there exists a unique canonical decomposition of \tilde{S} and it is \mathcal{L}_n^* -definable with parameters in $\langle \tilde{A} \rangle \cap \langle \tilde{B} \rangle$.

Let $m \in \mathbb{N}$ and I_1, \dots, I_m the multi-intervals such that $\bigcup_{j=1}^m I_j$ is the canonical decomposition of \tilde{S} . If $I_j = \bigcap_{i=1}^n (I_j^i \cap M)$, by Remark 4.2 and the definition of \tilde{A} and \tilde{B} , $I_j^i \cap M$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}$ -formula with parameters in $dcl_{\mathcal{L}_n}(A) \cap dcl_{\mathcal{L}_n}(B)$. □

Lemma 4.5. *Let M be a model of T and $e \in M^{eq}$. Let a be a tuple in M and f an $\mathcal{L}_n(\emptyset)$ -definable function such that $f(a) = e$. Let $E = \text{acl}(E) \supseteq \text{acl}^{eq}(e) \cap M$. Then there exist tuples b, b' in M , ACF-independent over E , such that $\text{tp}(b/E) = \text{tp}(b'/E) = \text{tp}(a/E)$ and $f(a) = f(b) = f(b')$.*

Proof. The proof is exactly the same as in claim 1 of Proposition 3.1 in [8]. If $a \in E$, it is clear. Assume $a \notin E$.

Sketch: Remember that by Fact 2.9 if $A \subseteq M$, then $\text{acl}(A) = \text{dcl}(A) = A^{\text{alg}} \cap M$. Using Neumann's Lemma we can find conjugates a_1, a_2 of a over $E \cup \{e\}$ satisfying:

$$\text{acl}(E, a_1) \cap \text{acl}(E, a_2) = E.$$

Take such a_1, a_2 with $\text{trdeg}(a_2/Ea_1) = m$ maximal satisfying (1) below

$$\text{tp}(a_1/E) = \text{tp}(a_2/E), \text{acl}(E, a_1) \cap \text{acl}(E, a_2) = E, \text{ and } f(a_1) = f(a_2) = e. \quad (1)$$

Take a_3 ACF -independent of a_2 over $E(a_1)$, such that $\text{tp}(a_3/Ea_1) = \text{tp}(a_2/Ea_1)$. Then $f(a_3) = f(a_2) = f(a_1) = e$ and $\text{acl}(E, a_1) \cap \text{acl}(E, a_3) = E$.

Since a_3 is ACF -independent of a_2 over $E(a_1)$, then $\text{acl}(E, a_3, a_1) \cap \text{acl}(E, a_2, a_1) = \text{acl}(E, a_1)$. Intersecting both sides with $\text{acl}(E, a_3)$ we obtain $\text{acl}(E, a_3) \cap \text{acl}(E, a_2, a_1) = E$ and then $\text{acl}(E, a_3) \cap \text{acl}(E, a_2) = E$.

Using the maximality of m , $\text{trdeg}(a_3/Ea_2) \leq m$, and so a_3 is ACF -independent of a_2 over $E(a_1)$ and it is also ACF -independent of a_1 over $E(a_2)$. Elimination of imaginaries in ACF and the fact that $E(a_1)^{\text{alg}} \cap E(a_2)^{\text{alg}} = E^{\text{alg}}$ imply that a_3 is ACF -independent of $a_1 a_2$ over E . Let $b = a_1$ and $b' = a_3$. \square

Lemma 4.6. *Let M be a sufficiently saturated model of T , $e \in M^{\text{eq}}$, $a \in M$ and f an $\mathcal{L}_n(\emptyset)$ -definable function such that $f(a) = e$. Let $E = \text{acl}^{\text{eq}}(e) \cap M$. Suppose that $e \notin \text{dcl}^{\text{eq}}(E)$.*

Then there is a multi-interval $I = \bigcap_{i=1}^n (I^i \cap M)$ such that $a \in I$ and $\{x \in I : \text{tp}(x/E) = \text{tp}(a/E) \wedge f(x) \neq e\}$ is multi-dense in I .

Proof. By Lemma 4.5 there exists $b \in M$, ACF -independent of a over E , such that $\text{tp}(b/E) = \text{tp}(a/E)$ and $f(a) = f(b)$. For each formula $\alpha(x) \in \text{tp}(a/E)$, define $\Phi_\alpha(x, y) := \alpha(x) \wedge f(x) \neq f(y)$. Then $\Phi_\alpha(M, a) = \Phi_\alpha(M, b) := A_\alpha$.

Since $e \notin \text{dcl}^{\text{eq}}(E)$, $\text{tp}(a/E)$ is not algebraic and $\text{tp}(a/E) \cup \{f(x) \neq e\}$ is consistent. Take $d \in M$ realizing $\text{tp}(a/E) \cup \{f(x) \neq e\}$. Then $d \in A_\alpha$.

Since A_α is $\mathcal{L}_n(Ea)$ -definable and also $\mathcal{L}_n(Eb)$ -definable, by Proposition 4.4 there exists a multi-interval $J_\alpha = \bigcap_{i=1}^n (J_\alpha^i \cap M)$ such that:

1. $d \in J_\alpha$
2. $\{x \in J_\alpha : x \in A_\alpha\}$ is multi-dense in J_α .
3. $J_\alpha^i \cap M$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}$ -formula with parameters in $\text{acl}(Ea) \cap \text{acl}(Eb) = E$.

So J_α is $\mathcal{L}_n(E)$ -definable in M . As $\text{tp}(d/E) = \text{tp}(a/E)$ and $d \in J_\alpha$, then $a \in J_\alpha$.

By saturation and Fact 2.5(1), there exists for all $i \in \{1, \dots, n\}$ an $<_i$ -interval I^i , with extremities in M such that $a \in I^i \subseteq \bigcap_{\alpha(x) \in \text{tp}(a/E)} (J_\alpha^i \cap M)$. Then $a \in I := \bigcap_{i=1}^n I^i$ and $\{x \in I : M \models \alpha(x) \wedge f(x) \neq e\}$ is multi-dense in I , for all $\alpha(x) \in \text{tp}(a/E)$. This implies using saturation that $\{x \in I : \text{tp}(x/E) = \text{tp}(a/E) \wedge f(x) \neq e\}$ is multi-dense in I . \square

Fact 4.7. [12, Theorem 3.21] Let $(M, <_1, \dots, <_n)$ be a model of T . Let $E = \text{acl}(E) \subseteq M$ and a_1, a_2, d tuples of M such that: d is ACF-independent of $\{a_1, a_2\}$ over E , $\text{tp}_{\mathcal{L}_n}(a_1/E) = \text{tp}_{\mathcal{L}_n}(a_2/E)$, and $\text{qftp}_{\mathcal{L}_n}(d, a_1/E) = \text{qftp}_{\mathcal{L}_n}(d, a_2/E)$. Suppose that $E(a_1)^{\text{alg}} \cap E(a_2)^{\text{alg}} = E^{\text{alg}}$.

Then there exists a tuple d^* in some elementary extension M^* of M such that:

- (1) d^* is ACF-independent of $\{a_1, a_2\}$ over E ,
- (2) $\text{tp}_{\mathcal{L}_n}(a_1, d^*/E) = \text{tp}_{\mathcal{L}_n}(a_2, d^*/E)$,
- (3) $\text{tp}_{\mathcal{L}_n}(a_1, d^*/E) = \text{tp}_{\mathcal{L}_n}(a_1, d/E)$.

Theorem 4.8. T has elimination of imaginaries.

Proof. Since we are working with a field it is enough to show that T has weak elimination of imaginaries. Let M be a monster model of T and $e \in M^{\text{eq}}$. Define $E := \text{acl}^{\text{eq}}(e) \cap M$. We need to show that $e \in \text{dcl}^{\text{eq}}(E)$. Let a be a tuple from M and let f be an $\mathcal{L}_n(\emptyset)$ -definable function such that $f(a) = e$. Suppose that $e \notin \text{dcl}^{\text{eq}}(E)$.

Claim 1. We can suppose that $\text{trdeg}(E(a)/E) = 1$:

Proof. Choose a with $\text{trdeg}(E(a)/E)$ minimal such that $f(a) = e$. Take $a' \subseteq a$ such that $\text{trdeg}(E(a)/E(a')) = 1$.

By Lemma 4.5 there is a tuple b in M , ACF-independent of a over E , such that $\text{tp}(a/E) = \text{tp}(b/E)$ and $f(a) = f(b)$.

Since b is ACF-independent of a over E and $a \notin \text{acl}(Ea')$, then $a \notin \text{acl}(Ea'b)$. As $e \in \text{dcl}^{\text{eq}}(b)$ then $\text{acl}^{\text{eq}}(Eea') \subseteq \text{acl}^{\text{eq}}(Ea'b)$. Thus $a \notin \text{acl}^{\text{eq}}(Eea')$. It follows without loss of generality that we can replace E by $\text{acl}(E(a'))$. \square

Suppose that $a = (a_1, \dots, a_m)$, $a_1 \notin E$ and $a \subseteq \text{acl}(Ea_1)$. Then $a_j \in \text{acl}(Ea_1) = \text{dcl}(Ea_1)$, for all $j \in \{1, \dots, m\}$, and we can suppose that $m = 1$.

By Lemma 4.5 there exists $b \in M$, ACF-independent of a over E , such that $\text{tp}(a/E) = \text{tp}(b/E)$ and $f(a) = f(b)$.

By Lemma 4.6 there is a multi-interval $I = \bigcap_{i=1}^n (I^i \cap M)$ such that $a \in I$, and $\{x \in I : \text{tp}(x/E) = \text{tp}(a/E) \wedge f(x) \neq e\}$ is multi-dense in I .

Claim 2. $\text{qftp}_{\mathcal{L}_n}(a/Eb) \cup \text{tp}_{\mathcal{L}_n}(a/E) \cup \{f(x) \neq e\}$ is consistent:

Proof. By compactness it is enough to show that if $\psi(x, b) \in \text{qftp}_{\mathcal{L}_n}(a/Eb)$, then there exists d such that $\text{tp}_{\mathcal{L}_n}(d/E) = \text{tp}_{\mathcal{L}_n}(a/E)$ and $M \models \psi(d, b) \wedge f(d) \neq e$.

As $\psi(x, b) \in \text{qftp}(a/Eb)$ and $a \notin \text{acl}(Eb)$, there is a multi-interval $J(b) := \bigcap_{i=1}^n (J^i(b) \cap M)$ such that $a \in J(b)$ and $J(b) \subseteq \psi(M, b)$.

As $a \in I^i \cap J^i(b)$ we can assume by taking the intersection that $J^i(b) \subseteq I^i$, for all $i \in \{1, \dots, n\}$. Since $\{x \in I : \text{tp}(x/E) = \text{tp}(a/E) \wedge f(x) \neq e\}$ is multi-dense in I we can find $d \in J(b)$, such that $f(d) \neq e$, and $\text{tp}(d/E) = \text{tp}(a/E)$. Then $M \models \psi(d, b) \wedge f(d) \neq e$. \square

Let d realize $\text{qftp}_{\mathcal{L}_n}(a/Eb) \cup \text{tp}_{\mathcal{L}_n}(a/E) \cup \{f(x) \neq e\}$. Then $\text{qftp}_{\mathcal{L}_n}(b, d/E) = \text{qftp}_{\mathcal{L}_n}(b, a/E)$, and $f(d) \neq f(b)$. By Fact 4.7 we can find b^* such that $\text{tp}_{\mathcal{L}_n}(b^*, d/E) = \text{tp}_{\mathcal{L}_n}(b^*, a/E)$ and $\text{tp}_{\mathcal{L}_n}(b^*, a/E) = \text{tp}_{\mathcal{L}_n}(b, a/E)$. As $f(b) = f(a) = e$, then $f(b^*) = f(a) = e$. But we have also that $f(b^*) = f(d) \neq e$. This is a contradiction. \square

Fact 4.9. [6, Theorem 4.12] *Any theory which has geometric elimination of imaginaries and for which algebraic closure defines a pregeometry is superrosy and $U^b(x = x) = 1$.*

Theorem 4.10. *Let M be a PRC bounded field and let $T = \text{Th}_{\mathcal{L}}(M)$ (see 2.8). Then T is superrosy and $U^b(x = x) = 1$.*

Proof. By Fact 2.9, algebraic closure defines a pregeometry and by Theorem 4.8 T has elimination of imaginaries. Then by Fact 4.9 T is superrosy and $U^b(x = x) = 1$. \square

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